# Dynamics of Ratchet Map

## ラチェット・マップのダイナミクス

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Abstract

We analyze the transport properties of a set of extensions of the Chirikov-Taylor Map which break both temporal and spatial reflection symmetries. We show that the simultaneous breaking of both symmetries result in the loss of periodicity in p-direction in the phase space dynamics, enabling the asymmetric diffusion which is the origin of the unidirectional ratchet motion. Replacement of the continuous dynamics by the discrete map enables us to look at the entire phase space structure with computational resources currently available, and even allows the calculation of the dynamical properties for the wide region in the controlling parameter space. We present the numerical results showing the intricate dependence of the asymmetric diffusion to the controlling parameters.

要旨

チリコフ＝テイラーの標準マップを、時間および空間反転不変性を破るように拡張することで「ラッチェット・マップ」というものが定式化できることを示す。この模型は空間的時間的に周期的なポテンシャルの中を運動する古典的一粒子を離散的な時間発展方程式によって記述するものである。通常の連続時間の発展方程式に比べ（1）見通しよく位相空間の運動の様子を把握分類できる（2）数値計算に適し現有の計算機資源でラチェット運動のコントロールパラメタ依存性を俯瞰する事ができる、という利点がある。そして実際に摩擦の無い場合、ある場合の両方でコントロールパラメタ空間上での単方向運動の微妙な存非の様子を示すグラフの一例を示す。

The motion of a particle in temporally and spatially periodic potential has attracted renewed attention as a model of molecular motors [2]. In order to make it a ratchet, i.e. a system which generates unidirectional motion out of periodic motion, one has to break both spatial and temporal reflection symmetry due to the Curie's theorem. Traditionally, the friction has supplied the source of temporal symmetry breaking, and it is still not clear in what condition the ratchet motion is brought about in "inertial" regime where the effect of friction is neglected. Most authors in the field base their conclusions on rather involved numerical results making them less than definite. Recent work by Flach *et al.* [3] has gone some way to clarify the situation with the careful analysis of the symmetry of the problem, but still suffers from the same problem.

In this note, we develop a model of a ratchet dynamics in which the time evolution is reducible to the discrete map instead of the differential equation. This reduces the computational burden substantially, and enables the analysis of the dynamics in the entire phase space. It also enables us to evaluate the transportation properties in an broad region in the controlling parameter space. We consider the motion of a classical particle described by the following time dependent Hamiltonian which has space periodic part and alternating current type driving motion;

$$H = \frac{p^2}{2} + \sum_{n=-\infty}^{\infty}\left[u(x) + xAs_n\right]\delta(t - nT\{1+\varepsilon s_n\}) \tag{1}$$

where we define

$$s_n = (-)^n \tag{2}$$

and

$$u(x) = -K\left[\cos x + \mu\cos(2x+\delta)\right] \ . \tag{3}$$

The evolution of the system is described by

$$\dot{x} = p \tag{4}$$

$$\dot{p} = \sum_{n=-\infty}^{\infty}\left[-u'(x) - As_n\right]\delta(t - nT\{1+\varepsilon s_n\}) - \gamma p \tag{5}$$

where we have added the friction term in Eq. (4) by hand to deal with damped motion. Because there are only delta-function kicks, one can integrate the motion in between the kickes and obtain the map which describe the evolution of position $x_n$ and momentum $p_n$ at discrete time $t_n = nT(1+\varepsilon s_n) - 0$

$$x_{n+1} = x_n + p_{n+1}(e^{\gamma(1+\varepsilon s_n)} - 1)/\gamma \tag{6}$$

$$p_{n+1} = e^{-\gamma(1+\varepsilon s_n)}[p_n - u'(x_n) - As_n] \tag{7}$$

with

$$u'(x) = K\{\sin x_n + 2\mu \sin(2x_n + \delta)\} \quad . \tag{8}$$

There are six parameters in the model: The parameter $K$ is the strength of the periodic potential that traps the particles in each periodic location, and $A$ is the amplitude of alternating swing motion. The parameters $\mu$ and $\delta$ are the measure of spatial asymmetry of the trapping potential and $\varepsilon$ is the measure of temporal reflection asymmetry of the alternating swing motion. The friction is controlled by its strength $\gamma$. Two limiting case is of particular interest. If one sets $\mu = \delta = 0$ and also $\gamma = 0$, one has

$$x_{n+1} = x_n + p_{n+1}(1+\varepsilon s_n) \tag{9}$$

$$p_{n+1} = p_n - K\sin x_n - As_n \tag{10}$$

One can regard this equation as a three-parameter extension of the standard map (Chirikov-Taylor map) which is of course the standard tool in the study of chaos as its name suggests [4]. Another useful limit is the case of $\varepsilon = 0$ which results in

$$x_{n+1} = x_n + p_{n+1}(e^{\gamma} - 1)/\gamma \tag{11}$$

$$p_{n+1} = e^{-\gamma}[p_n - u'(x_n) - As_n] \tag{12}$$

that describe the damped motion in spatially asymmetric trap potential plus alternating swing motion.

In this note, we mostly focus on frictionless case, Eqs. (9) and (10), leaving the detail of other case for the future publication [1] along with the full treatment of the model.

The identification of our model, Eqs. (9) and (10), as the extension of standard map immediately leads to the analysis of phase space: The transport property of standard map, $K = \varepsilon = 0$, has been analyzed in terms of phase space average of position and momentum $x$ and $p$. It is well known that their direct averages $<x>$ and $<p>$ are zero. However their mean square $<p^2>$ can be non zero if we have chaos unbounded by KAM tori in $p$-direction, which results in the chaotic diffusion which is characterized by the random-walk behaviour $<x^2>$ proportional to the time $t$. It is now obvious that the presence of the unidirectional ratchet motion is best characterized by the non-zero value for the $<p>$ itself. If one define $\tilde{x}_n \equiv x_n$, $\tilde{p}_n \equiv p_{n+1}$, one obtaines a reverse map

$$\tilde{x}_{n-1} = \tilde{x}_n - \tilde{p}_{n-1}(1-\varepsilon s_n) \tag{9'}$$

$$-\tilde{p}_{n-1} = -\tilde{p}_n - u'(\tilde{x}_n) - As_n \tag{10'}$$

If one has $\varepsilon = 0$, the map for $\{x_n, p_n\}$ and $\{\tilde{x}_n, \tilde{p}_n\}$ are identical, and one has

$$\langle -p \rangle = \langle -\tilde{p} \rangle = \langle p \rangle = 0 \tag{13}$$

Therefore, one needs both $A \neq 0$ and $\varepsilon \neq 0$ to have unidirectional motion. The situation is made visual in the following figures, Fig. 1.

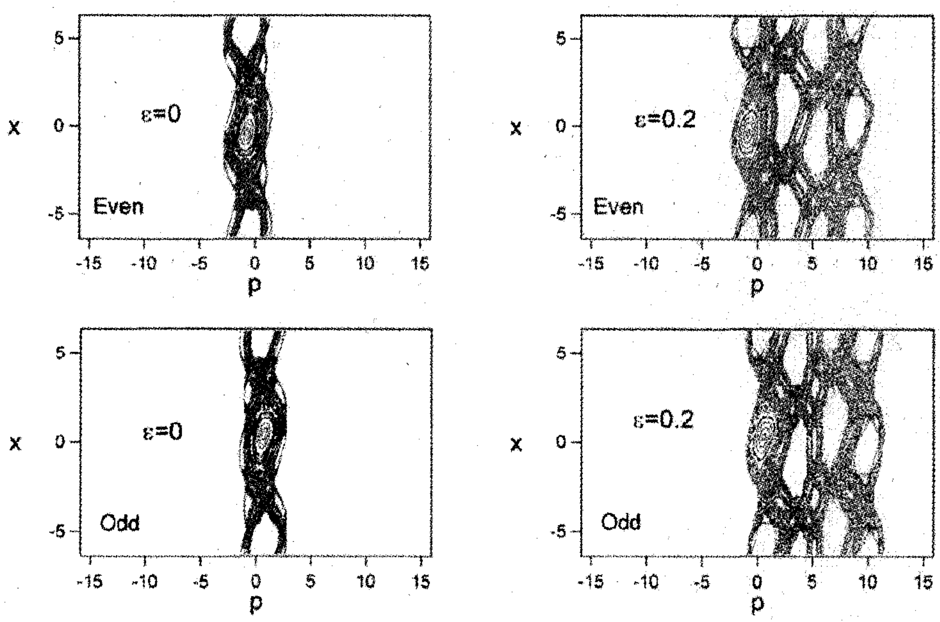


Fig.1 Phase space profile $p$ vs $x$. Left (a) A = 1.2 ε = 0, and right (b) A = 1.2 ε = 0.2. Other parameters are K = 0.6, μ = δ = 0.

In Fig 1, we depicted phase space profile for the ratchet map Eqs. (9) and (10) with (a) $A$ = 1.2, ε = 0 and (c) $A$ = 1.2, ε = 0.2. K is chosen to be 0.6. The graph is drawn by staring from a set of initial conditions clustered around $x$

= $p$ = 0. From these Figures, one clearly see the reason why and how one can have the unidirectional ratchet motion $<p> \neq 0$ in the case (b). There, because of the lack of symmetry with respect to the transformation $p \leftrightarrow -p$ in the face space, one can have chaotic region asymmetrically bounded by differently placed KAM tori in positive and negative $p$ regions. Thus we conclude that, in the frictionless case, the mechanism of unidirectional transport in the ratchet map Eq. (9) and (10) is the asymmetric chaotic diffusion. This is essentially in accordance with the conclusion of Flach *et al.* [3]., in which they call this phenomena "asymmetrical Levy flight".

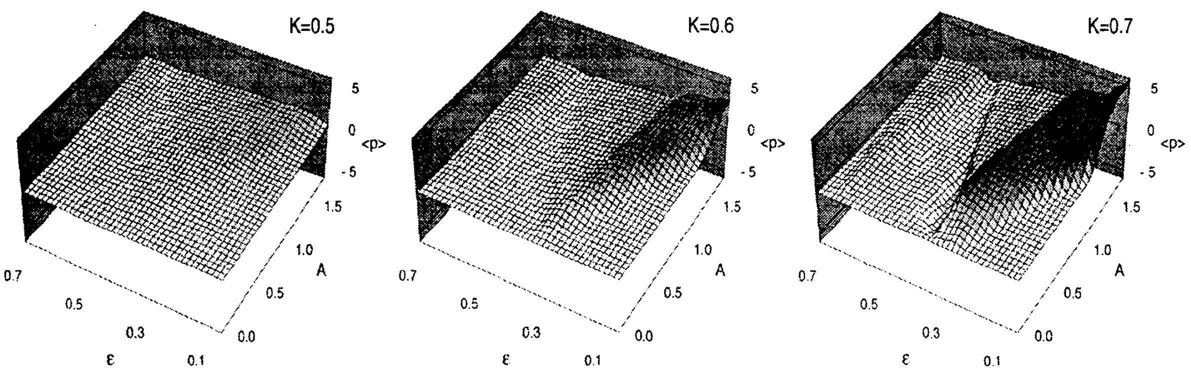


Fig.2 Surface plot ε, A vs <p>. From left to right, (a) K = 0.5, (b) K = 0.6 and (c) K=0.7. Other parameters are $\mu = \delta = 0$.

The advantage of our approach over the previous works including Ref [3] becomes clear in Fig. 2 where we plot $<p>$ as function of both parameters $A$ and $\varepsilon$ for frictionless case. This kind of plot would have been practically impossible in differential equation evolution without the reduction to the map. One can clearly see that the $<p>$ can be non-zero for non-zero $A$ and $\varepsilon$, and further more, the direction of the asymmetric transport (positivity or negativity) is controllable, although not predictable from the outset. Note also that the speed of unidirectional transport can be rather large: $<p>$ is almost $2\pi$ in the peak region, which means that the particle moves one spatial period in one temporal period!

In the closing of this note, let us simply flush the numerical results for the case with friction Eqs. (6) - (8). In Fig. 3, we plot $<p>$ as function of both parameters $A$ and $\gamma$. The parameter controlling the spatial asymmetry is set to be $\mu = 0.3$, $\delta = 0.2$.

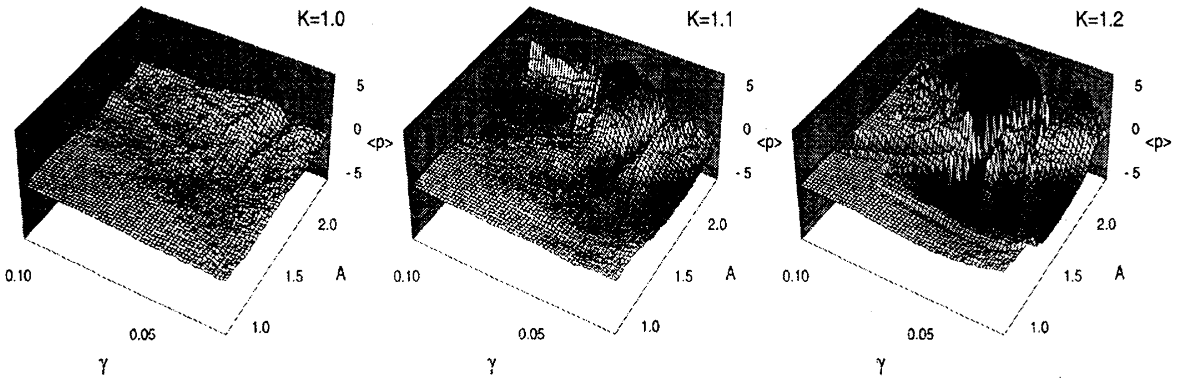


Fig.3 Surface plot $\gamma$, A vs <p>. From left to right, (a) $K = 1.0$, (b) $K = 1.1$ and (c) $K = 1.2$. Other parameters are $\mu = 0.3$, $\delta = 0.2$ and $\varepsilon = 0$.

Here again, we have unidirectional ratchet motion with sufficiently large $A$ and K once friction is turned on. The dependence of $<p>$ on control parameter, however, is very intricate in this case unlike the one in frictionless case. This is related to the existence of the strange attractors, which appears and disappears with the minute change of the controlling parameters.

In conclusion, we have formulated the ratchet map as an reflection symmetry breaking extension of standard map, and have shown some numerical examples that indicate its usefulness in the understanding of the dynamics of ratchet motions both with and without friction.

## Reference

The slide used in the talk (16 Nov, 2000 at YITP, Kyoto Univ.) can be found in http://www.mech.kochi-tech.ac.jp/cheon

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[4] See, for example, L.E. Reichl, “The transition to chaos: in conservative classical systems: quantum manifestations” (Springer, Heidelberg, 1991)